# Space optimal and asymptotically move optimal Arbitrary Pattern Formation on rectangular grid by asynchronous robot swarm<sup>\*</sup>

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Abstract. Arbitrary pattern formation (APF) is a well-studied problem in swarm robotics. The problem has been considered in two different settings so far; one is in a plane and another is in an infinite grid. This work deals with the problem in an infinite rectangular grid setting. The previous works in literature dealing with Apf problem in infinite grid had a fundamental issue. These deterministic algorithms use a lot of space in the grid to solve the problem mainly because of maintaining the asymmetry of the configuration or to avoid a collision. These solution techniques can not be useful if there is a space constraint in the application field. In this work, we consider luminous robots (with one light that can take two colors) to avoid symmetry, but we carefully designed a deterministic algorithm that solves the Apf problem using minimal required space in the grid. The robots are autonomous, identical, and anonymous and they operate in Look-Compute-Move cycles under a fully asynchronous scheduler. The Apf algorithm proposed in [WALCOM'2019] by Bose et al. can be modified using luminous robots so that it uses minimal space but that algorithm is not move-optimal. The algorithm proposed in this paper not only uses minimal space but also asymptotically move-optimal. The algorithm proposed in this work is designed for an infinite rectangular grid but it can be easily modified to work in a finite grid as well.

Keywords: Distributed computing · Arbitrary pattern formation · Rectangular grid · Robot with lights ·

## 1 Introduction

Swarm robotics in the field of a distributed system is well studied in the past two decades. Replacing a huge expensive robot with a set of simple, inexpensive robots is the goal of this field. This makes the system cost-effective, robust, and

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easily scalable. The robot swarm is usually modeled as a collection of computational entities, called robots, which can move. These robots operate in Look-Compute-Move (LCM) cycles. In the Look phase, a robot takes a snapshot of its surroundings as input. This input consists of the positions of other robots concerning its local coordinate system. In the Compute phase, the robot runs an inbuilt algorithm to determine a position to move. In the Move phase, the robot moves to that position. The main research interest has been to investigate what minimal capabilities are needed for these robots to solve a problem. The robots are assumed to be anonymous (robots have no unique identifiers), autonomous (there is no central control), homogeneous (all robots execute the same distributed algorithm), identical (the robots are indistinguishable from appearance), disoriented (the robots does not have access to a global coordinate). The robots can be oblivious, i.e., the robots do not have any memory to remember their past actions or past configuration. Each robot can be equipped with finite memory, where it can remember a finite bit. In literature, this model is termed as  $FSTA$ . Each robot can be able to communicate a finite bit of information to other robots. In literature, this model is termed as  $\mathcal{FCOM}$ . The finite bit memory and finite communicable information are together interpreted as a finite number of lights that can take a finite number of different colors. A robot with lights means the robot can access the color of the lights and it can communicate it to other robots. This model is termed as  $\mathcal{LUMI}$  and the robots are called luminous robots. Based on the timing to activation of the robots and execution time of the phases of the LCM cycles there are three types of schedulers in the literature. In the fully synchronous (FSync) scheduler, all robots operate in synchronous rounds. The time is divided into rounds. All robots simultaneously get activated and simultaneously execute the phases of the LCM cycle. In a semi-synchronous (SSync) scheduler, a nonempty set of robots gets activated in a round and simultaneously executes the phases of the LCM cycle. Next, in the fully asynchronous (ASync) scheduler, there is no common notion of time among the robots. All robots get activated and execute its LCM cycle independently.

The Arbitrary Pattern Formation (Apf) problem is one of the well-studied problems in the literature. This problem asks the robots to form a pattern that is given as input. the input is given as a set of points expressed in cartesian coordinates concerning a coordinate system. The goal of this problem is to design a distributed algorithm that allows a set of autonomous robots to form a specific but arbitrary geometric pattern given as input. This problem has been studied in both the euclidean plane and grid setting. In this paper, the problem is considered on an infinite rectangular grid for luminous robots in a fully asynchronous scheduler. The earlier solutions for this problem in grid settings did consider the space required for the solution. Infinite grid setting has theoretical motivation but practically one cannot have such luxury. For a space-constrained application field, we need an algorithm that uses lesser space. This will help to utilize the given space as optimally as possible. This somewhat guarantees lesser total robot movement as well. Motivated by this, this work proposes an algorithm that solves Apf problem in an infinite grid for asynchronous luminous robots. The proposed algorithm is space optimal and also asymptotically moves optimally. In the next section, we discuss the related works and contributions of this work.

## 2 Related work and our contribution

Related work The arbitrary pattern formation problem has been investigated mainly in two settings, the first one in a euclidean plane and another one in a grid. In euclidean plane the problem is studied in [\[3](#page-15-0)[,4,](#page-15-1)[5,](#page-15-2)[7,](#page-15-3)[8](#page-15-4)[,9,](#page-15-5)[16,](#page-16-0)[17\]](#page-16-1). In a grid setting, this problem is first studied in [\[2\]](#page-15-6). Here, the authors solved the problem in a rectangular grid by oblivious robots in ASync. Later in [\[6\]](#page-15-7), authors studied the problem on a regular tessellation graph. Whereas the algorithm proposed in [\[2](#page-15-6)[,6\]](#page-15-7) is not move optimal, so in [\[10\]](#page-15-8) authors provided two algorithms solving the problem in an asynchronous scheduler. The first algorithm solves the Apf problem for oblivious robots keeping the total robot move asymptotically optimal. The second algorithm solves the problem for luminous robots and this algorithm is asymptotically move-optimal and time-optimal. Then in [\[11\]](#page-15-9), the authors proposed two randomized algorithms solving the Apf problem in an asynchronous scheduler. The first algorithm works for oblivious robots. This algorithm is asymptotically move-optimal and time-optimal. The second algorithm works for luminous robots with obstructed visibility (when robots are not transparent). This algorithm is also move-optimal and time-optimal. In all the mentioned works for arbitrary pattern formation problems, firstly finding a solution was a challenge. Then the works tilted toward finding optimal solutions considering different aspects. So far, the considered aspects were the total number of moves made by the robots and the total time to solve the problem. None of the work mentioned or discussed the space complexity of the solution. In [\[12\]](#page-15-10), the author considered spatial complexity but they showed their solution is asymptotically space optimal. However, in the mutual visibility problem studied in [\[1,](#page-15-11)[15\]](#page-16-2) asymptotic space complexity has been considered.

Comparison with related works For the arbitrary pattern formation problem, in a rectangular grid, we define the space complexity of an algorithm as the minimum area of the rectangles (sides of which are parallel with the grid lines) such that no robot steps out of the rectangle throughout the execution of the algorithm. The work proposed in this paper is not only asymptotically space optimal (as in [\[12\]](#page-15-10)), it is exactly space optimal. Let the smallest enclosing rectangle (SER), the side of which are parallel to grid lines, of the initial configuration and pattern configuration formed by the robots respectively be  $m \times n$  ( $m \ge n$ ) and  $m' \times n'$  $(m' \geq n')$ , then minimum space required for an algorithm to solve the problem is a rectangle of dimension  $M \times N$ , where  $M = \max\{m, m'\}$  and  $N = \max\{n, n'\}.$ The deterministic algorithm proposed in this paper has space complexity  $M \times N$ , if  $M \neq N$  and has space complexity  $(M + 1) \times N$ , if  $M = N$ . The robots in this work only use one light that can take three different colors. Although, the algorithm proposed in [\[2\]](#page-15-6) can be modified such that it takes equal space as the algorithm of this work using luminous robots. But the proposed algorithm in [\[2\]](#page-15-6) is not move-optimal. The algorithms proposed in [\[10\]](#page-15-8) need the robots to form a compact line. The space complexity of these algorithms is  $M^2 \times N^2$  in the worst case. To the best of our knowledge, the work that is most closely related to our work is [\[11\]](#page-15-9). The first randomized algorithm, proposed in [\[11\]](#page-15-9) for luminous non-transparent robots, tends to use lesser space than all other existing works at this time. But this work did not discuss its space complexity. On investigating this work, authors find that this algorithm uses at least uses  $(M + 2) \times (N + 2)$ space to execute the algorithm. This algorithm also did not count the number of lights and colors required for the robots. The second randomized algorithm for oblivious robots in [\[11\]](#page-15-9) has space complexity  $10M \times 10N$ . Further, APF algorithms proposed in [\[14,](#page-16-3)[13\]](#page-16-4) solved it for obstructed visibility. These works also need to form a compact line, hence failing to be space optimal.

Our contribution This work gives first time a deterministic algorithm for solving APF in an infinite rectangular grid, which is space optimal as well as asymptotically move-optimal. Precisely, the space complexity for the algorithm is  $M \times N$ when  $M \neq N$  and  $(M + 1) \times N$  when  $M = N$ . And, if  $\mathcal{D} = \max\{M, N\}$ , then each robot requires to make  $O(D)$  moves. However, the algorithm can be easily modified to work for a finite grid that has enough space to contain both the initial and target configuration. The robots are asynchronous, luminous having one light that can take three different colors.

# 3 Model and problem statement

The robots are equipped with technology that a robot can determine other positions of other robots for a local coordinate system (chosen by the robot). The robots are on an infinite rectangular grid graph embedded in a plane. A robot chooses the local coordinate system such that the axes are parallel to the grid lines and the origin is its current position. Note that, robots do not agree on a global coordinate system. We consider that a robot can determine the position of all other robots present in the system for its local coordinate system irrespective of their position on the grid.

The robots are silent, i.e., they don't have any explicit way to communicate with each other. On activation, a robot executes a cycle, called the Look-Compute-Phase (LCM) cycle, which consists of three phases. In the Look phase, a robot takes a snapshot of its surroundings and gets the position and states of the robots which the robot communicates through implicit means. In Compute phase the robots run an inbuilt algorithm taking the information got in the Look phase and obtaining a state and position. The position can be its position or any of its adjacent grid nodes. In the Move phase, the robot changes its state as decided in Compute phase and either stay still or moves to the adjacent grid node as determined in Compute phase. The robots work asynchronously. There is no common notion of time for robots. Each robot independently gets activated and executes its LCM cycle. In this scheduler Compute phase and Move phase of robots take a significant amount of time. The time length of LCM cycles, Compute phases and Move Phases of robots may be different. Even the time length of two LCM cycles of one robot may be different. The gap between two consecutive LCM cycles or the time length of an LCM cycle of a robot is finite but can be unpredictably long. We consider the activation time and the time taken to complete an LCM cycle as determined by an adversary. In a fair adversarial scheduler, a robot gets activated infinitely often.

The robots are anonymous and indistinguishable from their appearance, which means, they don't have any unique identifier. All robots are autonomous, that is, they are not controlled by a central machine and run an inbuilt algorithm. The robots are homogeneous, which means, they all run the same distributed algorithm. Each robot has finite memory which is readable for other robots as well. The finite memory is interpreted as a finite number of lights that can take a finite number of colors. A robot can see another robot's lights and their present color.

Let  $\mathcal G$  be an infinite rectangular grid graph embedded on  $\mathbb R^2$ .  $\mathcal G$  can be formally defined as a geometric graph embedded on a plane as  $\mathcal{P} \times \mathcal{P}$ , which is the cartesian product of two infinite (from both end) path graphs  $P$ . Suppose a set of robots is placed on  $G$ . Let f be a function from the set of vertices of  $\mathcal G$  to  $\mathbb N \cup \{0\}$ , where  $f(v)$  is the number of robots on the vertex v of  $\mathcal G$ . Let g be a function from the set of edges of G to  $\mathbb{N} \cup \{0\}$ , where  $g(e)$  is the number of robots on the edge e of G. Then the pair  $(\mathcal{G}, f, g)$  is called a configuration of robots on G. We assume for the initial configuration  $(G, f)$ ,  $f(v) = 0$  or 1 for all node v in G and  $g(e) = 0$  for all edge e. Further, we assume the initial configuration to be asymmetric (See Subsection 3.2 of [\[2\]](#page-15-6) for the definition of asymmetric configuration).

Problem Statement Suppose a swarm of robots is placed in an infinite rectangle grid such that no two robots are on the same grid node and the configuration formed by the robots is asymmetric. The Arbitrary Pattern Formation (Apf) problem asks to design a distributed algorithm following which the robots autonomously can form any arbitrary but specific pattern, which is provided to the robots as an input, without scaling it or colliding with another robot.

# 4 The proposed algorithm

This section gives the proposed algorithm ApfMinSpace. We assumed that the initial configuration formed by the robots is asymmetric and all robot's lights have light color OFF. First, we describe a procedure named Procedure I, which can be executed by a robot if the configuration made by the robot is still asymmetric.

#### Procedure I:

Assumption: Configuration formed by the robots is still an asymmetric Let  $\mathcal C$  be the current configuration. Compute the smallest enclosing rectangle (SER) containing all the robots where the sides of the rectangle are parallel to the grid lines. Let  $\mathcal{R} = ABCD$  be the SER of the configuration, a  $m \times n$  rectangle with  $|AB| = n \ge m = |AD|$ . The length of the sides of R is considered as the number of grid points on that side. If all the robots are on a grid line then  $R$  is just a line segment. In this case, R is considered as  $1 \times n$  'rectangle' with  $A = D$ ,  $B = C$  and  $AD = BC = 1$ . Let  $n > m > 1$ , that is, R is a nonsquare rectangle. For each corner point A, B, C and D the robot calculates a binary string. For the corner point A, the binary string is determined as follows. Scan the grid from A along the longer side AB to B and sequentially all grid lines parallel to AB in the same direction. For each grid point, put a 0 or 1 according to whether it is empty or occupied by a robot. We denote the string as  $\lambda_{AB}$ . Similarly, the robot calculates other three strings  $\lambda_{BA}$ ,  $\lambda_{CD}$  and  $\lambda_{DC}$ . If  $\mathcal{R}$  is a square, that is,  $m = n$ , then we have to associate two strings to each corner. Then we have eight binary strings  $\lambda_{AB}$ ,  $\lambda_{BA}$ ,  $\lambda_{AD}$ ,  $\lambda_{DA}$ ,  $\lambda_{BC}$ ,  $\lambda_{CB}$ ,  $\lambda_{DC}$  and  $\lambda_{DC}$ . Since the configuration is asymmetric, so all the strings are distinct. the robot finds out the unique lexicographically largest string. Let  $\lambda_{AB}$  be the lexicographically largest string, then A is considered as the *leading corner* of the configuration. The leading corner is taken as the origin and  $\overrightarrow{AB}$  is as the x axis and  $\overrightarrow{AD}$  is as the y axis. If the R is a  $1 \times n$  rectangle then there are only two associated binary strings  $\lambda_{AB}$  and  $\lambda_{BA}$ . If both are equal then the configuration is symmetric. Since the configuration is asymmetric so the strings are distinct. Let  $\lambda_{AB}$  be the lexicographically largest string. Then A is considered as the origin and  $\overrightarrow{AB}$  is considered as the  $x$  axis. In this case, there is no common agreement of the  $Y$ axis. In all the cases, there is a unique string, say  $\lambda_{AB}$ . The robot responsible for the first 1 in this string is considered as *head* robot of  $C$  and the robot responsible for the last 1 is considered as tail of  $\mathcal C$ . The robot other than the head and tail is termed as inner robot.

Next, we discuss how robots are supposed to embed the target pattern when they agree on a global coordinate system. Let the  $\mathcal{R}' = A'B'C'D'$  be the SER of the target pattern, an  $m' \times n'$  rectangle with  $|A'B'| \geq |B'C'| > 1$ . We associate binary strings similarly for  $\mathcal{R}'$ . Let  $\lambda_{A'B'}$  be the lexicographically largest (may not be unique) among all other strings for  $\mathcal{R}'$ . The first target position on this string  $\lambda_{A'B'}$  is said to be *head target* and the last target position is said to be tail target. Then the target pattern is to be formed such that A' is the origin,  $\overrightarrow{A'B'}$ direction is along the positive x axis and  $\overrightarrow{A'D'}$  direction is along the positive y axis. Let the SER of the target pattern is a line  $A'B'$  and let  $\lambda_{A'B'}$  be a lexicographically largest string between  $\lambda_{A'B'}$  and  $\lambda_{B'A'}$ . Then the target is embedded in such a way that A' is at the origin and  $\overrightarrow{A'B'}$  direction is along the positive  $x$  axis.

Definition of head and tail robot Let formally state the definition of head and tail robot. Earlier in the Procedure, we defined the head and tail robot once. If there is no robot with TAIL color on, the configuration is still asymmetric, then that definition is applicable. If there is a robot with light color HEAD and there is another robot with a light color TAIL, then the robot with the HEAD color

on is said to be the head robot, and the robot with the TAIL color on is said to be a tail robot.

Next list some set of conditions in Table [1](#page-6-0) below. Before that state definitions. Let  $\mathcal{C}' = \mathcal{C} \setminus \{\text{head}\}\$ and  $\mathcal{C}'' = \mathcal{C} \setminus \{\text{head}, \text{tail}\}\$ , where  $\mathcal{C}$  is the current configuration. Let  $\mathcal{C}^{\prime}_{target} = \mathcal{C}_{target} \setminus \{\text{head target}\}, \ \mathcal{C}^{\prime \prime}_{target} = \mathcal{C}_{target} \setminus \mathcal{C}^{\prime \prime}_{target}$ {head target, tail target} where  $\mathcal{C}_{target}$  is the target configuration. Let the dimension of the SER of the current configuration be  $m \times n$  with  $m \leq n$  and the dimension of the SER of the target configuration be  $m' \times n'$  with  $m' \leq n'$ . If  $m \geq m'$  and  $n \geq n'$ , then the current SER can contain the target pattern.

$\overline{C_0}$	$\overline{\mathcal{C}} = \mathcal{C}_{target}$
$\overline{C_1}$	$\overline{\mathcal{C}'=\mathcal{C}'_{target}}$
$C_{2}$	$C'' = C''_{target}$
$\overline{C_3}$	Light color of each robot is OFF
$\overline{C_4}$	Light color of the head robot is HEAD and the light color
	of the rest robots is OFF
$C_5$	There is a robot with light color HEAD and there is a
	robot with light color TAIL
$ C_6 $	The tail robot is at a corner point if the SER
$\overline{C_7}$	The current SER can contain the target pattern
$\overline{C_8}$	The current SER is a nonsquare rectangle

<span id="page-6-0"></span>Table 1. List of conditions

Let's formally describe the algorithm. There are three different phases. Phase I runs when  $C_3$  is true for the current configuration, phase II runs when  $C_4$  is true for the current configuration and phase III runs when  $C_5$  is true for the current configuration. Next, we discuss the details of the phases one by one.

#### Phase I

*Condition:*  $C_3$  is true.

In this phase, if in the visible configuration in the snapshot a robot is seen on an edge then discard the snapshot and go to sleep. If the current configuration is symmetric, then terminate. Else if the configuration is asymmetric, then run Procedure I and determine the global coordinate system. If  $C_0$  is true, then terminate. If  $\neg C_0 \wedge C_1$  is true, then the head robot goes to the left (right) if the head target is it's left (right). If  $\neg C_0 \wedge \neg C_1$  is true, then the head robot moves to the left until it reaches the origin. If the head robot is at its origin then change the color of its light to HEAD.

*Output:*  $(C_3 \wedge C_0) \vee (C_4 \wedge \neg C_1)$  is true.

## Phase II

*Condition:*  $C_4$  is true.

If  $C_0$  is true then the head robot turns its light color to OFF. If  $C_0$  is not true and the configuration is asymmetric then run Procedure I. If  $C_1$  is true then the head robot turns its light color OFF. Else, find the tail robot to turn its light color as TAIL.

*Output:*  $(C_3 \wedge (C_0 \vee C_1)) \vee (C_5 \wedge \neg C_1)$  is true.

Next, first we describe another procedure, Procedure II before describing phase III.

Procedure II:

Assumption:  $C_5$  is true.

Let SER of the configuration be a rectangle  $ABCD$  with  $|AB| \geq |BC|$  and head robot situated at A.

Case-I: Let ABCD be a nonsquare rectangle or, ABCD be a square and the tail robot is on the CD edge but not at C. Then consider A as origin,  $\overrightarrow{AB}$  as x axis and  $\overrightarrow{AD}$  as y axis.

Case-II: If  $ABCD$  is a square rectangle and the tail robot is at  $C$ , then consider A as the origin and there are two possibilities of consideration of axes. Firstly, it can be done by considering  $\overrightarrow{AB}$  as x axis and  $\overrightarrow{AD}$  as y axis. Secondly, it can be done by considering  $\overrightarrow{AD}$  as x axis and  $\overrightarrow{AB}$  as y axis.

Let the SER of the configuration be a line  $AB$  and the head robot situated as  $A$ . Then the tail robot is situated at  $B$ . Then consider  $A$  as the origin and  $\overrightarrow{AB}$  as x axis. The y axis can be considered in either way out of the two possible ways.

### Phase III

*Condition:*  $C_5$  is true.

In this phase, if in the snapshot, the tail robot is seen on the edge then discard the snapshot and go to sleep. Execute Procedure II. If considering the coordinate system through case I or any of the coordinate systems through case II  $C_2$  is true, then the tail moves towards the tail target and upon reaching the tail target turns its light color to OFF. Else if  $C_2$  is not true considering any of the coordinate systems through Procedure II, if  $C_6$  is false then the tail robot moves right. If  $\neg C_2 \wedge C_6$  is true then there are two possibilities; either  $C_7$  is true or not. If  $C_7$  is not true then the tail robot expands the SER to fit the target pattern. If  $\neg C_2 \land C_6 \land C_7$  is true but  $C_8$  is false, then the tail robot moves outside the SER. Finally, when  $\neg C_2 \wedge C_6 \wedge C_7 \wedge C_8$  is true, then call function Rearrange().

## Function Rearrange()

Input:  $\neg C_2 \wedge C_6 \wedge C_7 \wedge C_8$  is true.

Let's name the grid lines parallel to  $x$  axis (we shall call them horizontal grid lines)  $H_1, H_2, \ldots$ , from down to top. Let the horizontal line containing the head is  $H_1$ . Let  $a'(i)$   $(b'(i))$  is the total number of target positions in  $\mathcal{C}'_{target}$  above (below)  $H_i$  horizontal line. Let  $a(i)$   $(b(i))$  is the total number of robots in  $\mathcal{C}_{target}$ above (below)  $H_i$  horizontal line. We say a horizontal line  $H_i$  satisfies upward condition if:

(U1)  $a'(i) > a(i)$ , (U2)  $[a'(i+1) > a(i+1)$  and  $H_{i+1}$  is empty] or  $[a'(i+1) = a(i+1)].$ 

Then we say a horizontal line  $H_i$  satisfies *downward* condition if:

 $(D1)$   $b'(i) > b(i),$ (D2)  $[H_{i-1} \text{ is empty}] \text{ or } [b'(i-1) \leq b(i-1)].$ 

Suppose a horizontal line  $H_i$  satisfies the upward condition but not the downward condition. If there is a robot in between  $H_i$  and  $H_{i+1}$  then no robot on  $H_i$  does anything. Let there be no robot in between  $H_i$  and  $H_{i+1}$ . If  $H_{i+1}$  is empty then the leftmost robot on  $H_i$  goes upward. If  $H_{i+1}$  is nonempty and there is a robot on  $H_i$  which has its upward node empty then the leftmost such robot goes upward. If there are no such robots that have no upward node empty, then consider the leftmost empty node v on  $H_{i+1}$  in the SER. Then consider the closest robot on  $H_i$  to v node. If there are two such robots then the left one moves to  $v$ .

Suppose a horizontal line  $H_i$  satisfies the downward condition. If there is a robot in between  $H_i$  and  $H_{i-1}$  then no robot on  $H_i$  does anything. Let there be no robot in between  $H_i$  and  $H_{i-1}$ . If  $H_{i-1}$  is empty then the rightmost robot on  $H_i$  goes downward. If  $H_{i-1}$  is nonempty and there is a robot on  $H_i$  which has its downward node empty then the rightmost such robot goes downward. If there are no such robots that have no downward node empty, then consider the rightmost empty node v on  $H_{i-1}$  in the SER. Then consider the closest robot on  $H_i$  to v node. If there are two such robots then the left one moves to v.

A horizontal line  $H_i$  is said to be *saturated* if  $a(i) = a'(i)$  and  $b(i) = b'(i)$ . Suppose a robot, r on a saturated horizontal line, say  $H_i$ . If r is the  $j<sup>th</sup>$  robot from left on  $H_i$ , then consider the  $j<sup>th</sup>$  target position at a node, say u, on the  $H_i$ from the left. If the  $u$  node is at the left (right) of  $r$  and the left (right) neighbor node of  $r$  is empty then move left (right).

Aim of Rearrange:  $C_2 \wedge C_5$  is true.

Output:  $C_1 \wedge C_4$ 

For better understanding phase III is illustrated in the flowchart depicted in Figure [1.](#page-9-0) Next, the flow of the algorithm is given in the flowchart depicted in Figure [2.](#page-10-0) In the next section, we prove the correctness of the proposed algorithm. The target of the algorithm is to achieve  $C_0 \wedge C_3$ .

# 5 Correctness of the proposed algorithm

We start by proving the correctness of the three phases.

#### Correctness of Phase I

The algorithm enters in phase I when  $C_3$  is true. There are two sub-cases either  $C_1$  is true or not. If in the initial configuration,  $C_1$  is not true then the head robot moves left until it reaches the leading corner. Let the SER of the configuration be ABCD at some time such that  $\lambda_{AB}$  is the lexicographically largest string. If the head moves left then the  $\lambda_{AB}^{new}$  is lexicographically larger than  $\lambda_{AB}$  and also than the other considered strings. Therefore, the new configuration is still asymmetric and the coordinate system remains unchanged. Thus after a finite



<span id="page-9-0"></span>Fig. 1. Illustration of Phase III

number of moves to the left head reaches its origin and turns its HEAD color on. This makes  $\neg C_1 \land C_4$  true. If in the initial configuration  $C_1$  is true. If in the initial configuration  $C_0$  is not true then the configuration must be asymmetric. Then the head robot can be identified through Procedure I. In such a case head approaches to head target. Since at the beginning of this phase head is at the origin so either the head target is at the origin or it is at the right of the head. If  $C_0$  is not true, the head target is at the left of the head. Let at some time t head at a position  $h$  and it moves left. If after this move head reaches the head target then  $C_0 \wedge C_3$  becomes true. Otherwise, we show after this move we show that the configuration remains asymmetric and the coordinate system remains unchanged. Suppose at this time the SER of the current configuration is ABCD then as directed in the algorithm the SER of the embedded target pattern should be also  $ABCD$  with  $\lambda_{AB}$  as a lexicographically largest string and position of the head target is the first 1 in  $\lambda_{AB}$ . Until the head reaches head target, it remains at the left of the head target. So the current configuration remains asymmetric and the  $\lambda_{AB}$  string for the current configuration remains lexicographically largest. Therefore, after finite time head reaches at head target making  $C_0 \wedge C_3$  true. We conclude the above discussion in the following Lemma [1.](#page-9-1)

<span id="page-9-1"></span>**Lemma 1.** After finite time execution of phase I,  $(C_3 \wedge C_0) \vee (C_4 \wedge \neg C_1)$  becomes true.

#### Correctness of Phase II

In this phase, no robots are directed to move. The correctness of this phase is basic. We conclude it in the following Lemma [2.](#page-9-2)

<span id="page-9-2"></span>**Lemma 2.** After finite time execution of phase II,  $(C_3 \wedge (C_0 \vee C_1)) \vee (C_5 \wedge \neg C_1)$ becomes true.



<span id="page-10-0"></span>Fig. 2. Flow Chart of the Algorithm

\*Here in Phase I, only the head robot moves left to reach the origin which does not create symmetry in the configuration

\*\* $C_1$  is true when the head robot is at the origin (from  $C_4$ ) and the configuration is symmetric, this implies the head target is at the origin, which makes  $C_0$  true.

## Correctness of Phase III

In this phase, the head robot is at a corner with its HEAD color on, and the tail robot is on an edge with a TAIL color on. First, we proof the correctness of the Rearrange function. Condition of this function is  $C_5 \wedge C_6 \wedge C_7 \wedge C_8$ . Since only inner robots are moving in this function,  $C_7$  remains true if only inner robots move. Next in this function, no inner robot is allowed step out of the SER formed by the head and tail robot, so  $C_6$  and  $C_8$  also remain true. So coordinate system decided through Procedure II also remains unchanged throughout. Now once a horizontal line becomes saturated then after a finite time all robots on that line take their respective target position by horizontal moves. So it is sufficient to show that, every horizontal line becomes saturated after a finite time. Therefore, we get the liberty to throw away the case when the initial and target configuration both have the SER, a line, because in that case there is only one horizontal line which is saturated vacuously according to the definition.

Let there be some nonsaturated horizontal lines in the configuration at some time. Let us consider two consecutive saturated horizontal lines,  $H_i$  and  $H_j$  such that  $|i - j| \neq 1$ . If there is no saturated horizontal line or only one saturated horizontal line, then consider this scenario in the following way. Let the SER of the configuration ABCD where head and tail are respectively situated at A and  $C$ . Then consider the horizontal line below the  $AB$  and the horizontal line above CD. We can consider these two lines as vacuous saturated lines. The scheme of the proof is, we show that after a finite time, another saturated horizontal line will create between the lines  $H_i$  and  $H_j$ . Without loss of generality, let  $i > j$ . Note that, there have to be at least two horizontal lines between  $H_i$  and  $H_i$ . Consider the horizontal line  $H_{i-1}$ . Note that,  $H_{i-1}$  cannot satisfy upward condition because  $a'(i-1) = a(i-1)$ . According to the assumption,  $H_{i-1}$  is not a saturated horizontal line. So we must have  $b'(i-1) > b(i-1)$  or  $b'(i-1) < b(i-1)$ .

Case I:  $(b'(i-1) > b(i-1))$  Let starting from  $H_{i-1}$  and going downwards  $H_k$  is the last horizontal line such that  $b'(k) > b(k)$  and  $k > j + 1$ . Then  $b'(k-1) \leq b(k-1)$  and  $b'(p) > b(p)$  for all  $p = i-1, i-2, \ldots, k$ . The existence of such a horizontal line is guaranteed because  $b'(j+1) = b(j+1)$ . Consider the horizontal line  $H_k$ . Then  $H_k$  satisfies the downward condition. If  $H_k$  is nonempty then a robot will come down.  $H_k$  horizontal line will keep satisfying downward condition until  $b'(k) = b(k)$  becomes true. Suppose  $H_k$  is empty, then consider the first nonempty horizontal line  $H_m$  above  $H_k$ . Then  $H_m$  satisfies the downward condition. Then a robot comes down from  $H_m$ . That robot comes down to  $H_k$  making  $H_k$  nonempty. Hence after finite time  $b'(k) = b(k)$  becomes true. Now if at this time  $b'(k+1) = b(k+1)$  then  $H_k$  is saturated and our task is done. Since only robots were coming down through  $H_{k+1}$  in this time interval, therefore  $H_{k+1}$  was satisfying (D1) so the difference  $b'(k+1) - b(k+1)$  can minimum reach to zero. So we have only remaining possibility at this time, that is,  $b'(k+1) > b(k+1)$ . Suppose  $b'(k+1) > b(k+1)$ , then now  $H_{k+1}$  satisfies the downward condition. And similarly, after finite time  $b'(k+1) = b(k+1)$  becomes true and which implies  $H_k$  is saturated.

Case II:  $(b'(i-1) < b(i-1))$  For this case we have  $a'(i-2) > a(i-2)$  and we have  $a'(i-1) = a(i-1)$ . Hence  $H_{i-2}$  satisfies the upward condition. If  $H_{i-2}$  also satisfies the downward condition then  $H_{i-2}$  must be nonempty then after finite sometime required robot(s) will go down from  $H_{i-2}$ , making it no longer satisfy the downward condition. We assume  $H_{i-2}$  satisfies the upward condition but not the downward condition. If  $H_{i-2}$  is nonempty then a robot goes upward and reaches  $H_{i-1}$ . The  $H_{i-2}$  will keep satisfying the upward condition and the robot will keep coming up from  $H_{i-2}$  until  $H_{i-2}$  is empty or  $a'(i-2) = a(i-2)$ . If  $a'(i-2) = a(i-2)$  turns true then  $H_{i-1}$  becomes saturated. If  $a'(i-2) > a(i-2)$ is true and  $H_{i-2}$  is empty. Then consider the first nonempty horizontal line  $H_m$  below  $H_{i-1}$ . Note that such a nonempty line must exist. Consider  $H_{i-3}$ horizontal line. We have  $a'(i-2) > a(i-2)$  is true and  $H_{i-2}$  is empty. This forces to satisfy  $a'(i-3) > a(i-3)$ . So,  $H_{i-3}$  satisfies the upward condition. Similarly, we can show that all horizontal lines  $H_{i-2}, \ldots, H_m$  satisfy the upward condition. Since  $H_m$  satisfies the upward condition, a robot comes upward from  $H_m$ . And that robot reaches  $H_{i-2}$  making  $H_{i-2}$  non empty. Hence after finite time  $H_{i-1}$  becomes saturated.

The rest four types of executions by the tail robot are quite basic. Hence we conclude it in the following Lemma [3.](#page-12-0)

<span id="page-12-0"></span>**Lemma 3.** After finite time execution of phase III,  $C_4 \wedge C_1$  becomes true.

Next, we prove the correctness of the proposed algorithm. The goal of our algorithm is to make a configuration where  $C_3 \wedge C_0$  is true. If the initial configuration doesn't match with the target configuration then  $\neg C_0 \wedge C_3$  is true. The flow chart depicted in Figure [2](#page-10-0) shows that any directed path starting from  $\neg C_0 \wedge C_3$  ends at  $C_0 \wedge C_3$  passing through the phases finite times. Hence the correctness follows.

Theorem 1. The algorithm APFMINSPACE solves the APF problem within finite time.

# 6 Space complexity of the proposed algorithm

In this section, we calculate the maximum space required for the robots to execute the algorithm ApfMinSpace. In Phase, I head maximum reaches the leading corner of the current SER, which gives the head robot never step out of the current SER. The inner robot only moves in Rearrange function where also they are not allowed to step out of the current SER. Now the tail robot only steps out of the current SER if it has to expand the SER to contain the target pattern. Hence if the SER of the initial configuration can contain the target pattern then no robot steps out of the SER of the current configuration. Otherwise, the tail expands the SER exactly to fit the target pattern. Hence, the robots only move inside a rectangle with minimum dimensions which contains both initial and target configuration. Precisely, if  $m \times n$   $(m \ge n)$  and  $m' \times n'$   $(m' \ge n')$  are the dimensions of the SER of the initial configuration and target configuration respectively, then the robots only move inside a rectangle of dimension  $M \times N$ where  $M = \max\{m, m'\}$  and  $N = \max\{n, n'\}$ . Further, if  $M = N$  then the tail moves one step away from the current SER to make the SER non square in phase III. Hence we record the maximum required space in the following Theorem [2.](#page-12-1)

<span id="page-12-1"></span>**Theorem 2.** If  $m \times n$  ( $m \ge n$ ) and  $m' \times n'$  ( $m' \ge n'$ ) are the dimensions of the SER of the initial configuration and target configuration respectively. Let  $M = \max\{m, m'\}$  and  $N = \max\{n, n'\}$ . Then throughout the execution of algorithm APFMINSPACE, the robots are only required to move inside a rectangle of dimension  $M \times N$  or  $(M + 1) \times N$  in accordance with  $M > N$  or  $M = N$ .

## 7 Move complexity of the proposed algorithm

In this section, we show that the each robot makes  $O(D)$  moves throughout the algorithm, where  $D$  is the dimension of minimum square which can contain both initial and target configuration. Which will show that the proposed algorithm is asymptotically move optimal. First, we consider the movements of the head and tail robots. The head robot only moves through the  $x$  axis and its maximum locus is from initial position to origin and then origin to head target. Hence, head maximum makes 2D moves. The tail robot might change its horizontal line but only for once then it moves to tail target. Therefore, tail makes at most 2D moves. Next, let  $r$  be an inner robot initially belonged to the horizontal line  $H_i$ . If  $H_i$  is saturated then r makes maximum D moves to settle down.

Suppose  $H_i$  is not saturated initially. Before that we draw an observation. If for a horizontal line  $H_i$ ,  $a'(i) \leq a(i)$   $(b'(i) \leq b(i))$ , then no robot ever goes upward (downward) from  $H_i$ . If  $a'(i) \leq a(i)$  is true then it implies and implied by  $b'(i+1) \ge b(i+1)$ . If  $a'(i) = a(i)$  then it implies and implied by  $b'(i+1) = b(i+1)$ . So at this condition neither  $H_i$  satisfies upward condition nor  $H_{i+1}$  satisfies downward condition. So, there will be no exchange of robots in between this two horizontal lines. Suppose  $a'(i) < a(i)$ . Then this implies and implied by  $b'(i+1) > b(i+1)$ . Then eventually leads to satisfy the downward condition for  $H_{i+1}$ . Let  $l = b'(i+1) - b(i+1)$ , then from the proposed algorithm a unique robot fixed robot on  $H_{i+1}$  robot comes down from  $H_{i+1}$  to make the difference  $b'(i + 1) - b(i + 1) = l - 1$ . If  $l - 1 > 0$ , there is another fixed unique robot which comes down from  $H_{i+1}$  making the difference  $l-2$ . Hence, eventually the difference  $b'(i + 1) - b(i + 1)$  becomes zero. After this no exchange of robots between the horizontal lines  $H_i$  and  $H_{i+1}$  takes place. Thus, if  $a'(i) \leq a(i)$  is true, then no robot ever goes upward from  $H_i$ . Similarly, one can show that, if for a horizontal line  $H_i$ ,  $b'(i) \leq b(i)$ , then no robot ever goes downward from  $H_i$ .

Thus, if  $a'(i) \leq a(i)$  and  $b'(i) \leq b(i)$ , r never leaves the  $H_i$ . Eventually  $H_i$ gets saturated, so in this case also r makes at most D moves. Otherwise, after some time either it satisfies upward condition or downward condition or both. In this case also either r never leaves the horizontal line or  $r$  goes upward or downward. Suppose r goes upward, then at that time  $a'(i) > a(i)$  and either  $H_{i+1}$  is empty or  $a'(i+1) = a(i+1)$ . First we show that if  $a'(i+1) = a(i+1)$ is true then r never leaves  $H_{i+1}$  after reaching there. If  $a'(i+1) = a(i+1)$ , then from previous discussion no robot ever goes up from  $H_{i+1}$ . After r moves upward if  $a'(i) = a(i)$  becomes true then  $H_{i+1}$  is saturated. Otherwise, if  $a'(i) > a(i)$ remains true, then  $b'(i+1) < b(i+1)$  then  $H_{i+1}$  does not satisfy the downward condition. Hence  $H_{i+1}$  does not satisfy either upward or downward condition. So then r does not leave the  $H_{i+1}$  after that.

Next, suppose  $H_{i+1}$  is empty and  $a'(i+1) > a(i+1)$ . If r moves upward then it just makes one vertical movement to reach  $H_{i+1}$ . Similarly, if after reaching  $a'(i) = a(i)$  becomes true then  $H_{i+1}$  becomes saturated. Hence we conclude if r goes upward from  $H_i$  under the condition  $a'(i + 1) = a(i + 1)$  then it r settles down at a target position on  $H_{i+1}$ . And if r goes upward from  $H_i$  under the condition that  $H_{i+1}$  is empty then it just makes a vertical movement to reach  $H_{i+1}$ . A similar conclusion can be made similarly if r starts moving downwards in the first place.

Next, we show that throughout the execution of the algorithm if  $r$  starts moving upward then it never comes down after that, and also, if it starts moving downwards initially then it never goes upward after that. On the contrary, let the opposite happens. Then without loss of generality, there exists  $i$  such that  $r$ goes upward from  $H_i$  to  $H_{i+1}$  and then after some time again comes down. r goes upward implies  $a'(i) > a(i)$  and any other robot can go upward after that only if  $a'(i) > a(i)$  remains true. After a robot goes upward we must have  $a'(i) \geq a(i)$ . That implies  $b'(i+1) \leq b(i+1)$ , but with this  $H_{i+1}$  can never satisfy downward condition. So no robot can come downwards from  $H_{i+1}$ . Summarising all if a robot r starts initially going upward its locus would be a vertical movement in a straight line until reaches a horizontal line  $H_i$  where  $a'(i) = a(i)$ , this takes at most  $D$  moves. Then  $r$  maximum makes  $D$  moves on the  $H_i$  horizontal line before moving to  $h_{i+1}$ . Then r moves upward on  $H_{i+1}$  and settles down at a target node on  $H_{i+1}$ , which also takes at most D moves. Hence r all total makes  $3D$  moves. Similarly, one can show if r starts moving downwards initially then also r all total makes 3D moves.

<span id="page-14-0"></span>**Theorem 3.** The algorithm APFMINSPACE requires each robot to make  $O(D)$ moves.

## 8 Conclusion

This work provides an algorithm for solving arbitrary pattern formation problems by robot swarm. The robots are considered autonomous, anonymous, and identical. The proposed algorithm works for asynchronous robots with one light that can take three different colors. The algorithm uses minimal space to solve the Apf problem (Theorem [2\)](#page-12-1). Further, the algorithm is asymptotically moveoptimal (Theorem [3\)](#page-14-0). Even though the proposed algorithm is considered over an infinite rectangular grid, the algorithm can be easily modified to work in a finite rectangular grid (This part can be seen in a detailed version of the work).

This work does not investigate (due to space constrain) whether the algorithm is asymptotically time optimal or not. If the proposed algorithm is not time optimal then it would be interesting to find whether there exists an algorithm that is asymptotically move-optimal, time-optimal, and also space optimal. Further, in the proposed algorithm, for a case when  $M = N$ , the algorithm required the space  $(M + 1) \times N$ . The authors do not know whether this can be improved to  $M \times N$ . Fixing this issue does not contribute a lot but it is under process. Even though the proposed algorithm is asymptotically move-optimal but the authors believe that the total required move is better than existing move optimal APF algorithms (shall be investigated in a detailed version).

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